Integrali di superficie

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Analisi II

Richiami di teoria

- Sia $\mathcal{S} = \vec{r}(T)$ una superficie parametrica in $\mathbb{R}^3$, ove $T \subset \mathbb{R}^2$ è connesso e limitato e
  $\vec{r} : T \to \mathbb{R}^3$, $\vec{r} \in C^1(T)$, è data da:
  $$\vec{r}(u,v) = x(u,v)\vec{i}_1 + y(u,v)\vec{i}_2 + z(u,v)\vec{i}_3\,, \ \ (u,v) \in T$$
- Sia $g : \mathcal{S} \to \mathbb{R}$ limitata su $\mathcal{S}$, $g = g(x,y,z)$

Si definisce l'integrale di superficie di g su $\mathcal{S}$:

$$\iint_{\mathcal{S}} g(x,y,z)\, d\mathcal{S} := \iint_T g(x(u,v), y(u,v), z(u,v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dudv$$

$$\text{con} \quad \begin{cases} \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial x}{\partial u}\vec{i}_1 + \dfrac{\partial y}{\partial u}\vec{i}_2 + \dfrac{\partial z}{\partial u}\vec{i}_3 \\[2ex] \dfrac{\partial \vec{r}}{\partial v} = \dfrac{\partial x}{\partial v}\vec{i}_1 + \dfrac{\partial y}{\partial v}\vec{i}_2 + \dfrac{\partial z}{\partial v}\vec{i}_3 \\[2ex] \underbrace{\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}}_{\text{prodotto vettoriale di } \frac{\partial \vec{r}}{\partial u} \text{ e } \frac{\partial \vec{r}}{\partial v}} = \det \begin{bmatrix} \vec{i}_1 & \vec{i}_2 & \vec{i}_3 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \end{cases}$$

In particolare:

$$\mathrm{Area}(\mathcal{S}) = \iint_{\mathcal{S}} d\mathcal{S} = \iint_{T} \left\| \frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v} \right\| dudv$$

Se $\mathcal{S}$ è data in <u>forma cartesiana</u>:

$$z = f(x,y)\,, \quad (x,y) \in T\,,$$

allora:

$$\iint_{\mathcal{S}} g(x,y,z)\, d\mathcal{S} = \iint_{T} g(x,y,f(x,y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$$

e

$$\mathrm{Area}(\mathcal{S}) = \iint_{T} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$$